Lie Group Analysis in Object Recognition

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Abstract

The techniques of Lie group analysis can be used to determine absolute invariant functions which serve as classifier functions in object recognition problems. Lie group analysis is a powerful tool for analyzing complex systems such as the conservation model used in recent thermophysical invariance (TPI) research. We will discuss the mathematics of Lie groups and the application to recognition problems (TPI specifically). The experimental results will demonstrate the validity of the methods and determine the direction of future research. More extensive background and results are available in an extended version of this paper.

1 Introduction

In a nutshell here's what these techniques provide and how they can be used in classifying objects: Lie group analysis will determine if there exists a non-trivial function Φ which assumes a constant value on the set of all roots of an equation $f(\vec{z}) = 0$. The form of the equation remains constant regardless of which particular object we are measuring (viewing), but some of the coefficients in this equation may (and generally will) change depending upon the object being viewed, as for example when $f(\vec{z}) = 0$ expresses a conservation equation. As a result, the set of roots will differ depending upon the object being viewed. Correspondingly the constant value $\Phi(\vec{z})$ will assume a different value depending upon the object being viewed, thus permitting the use of Φ as a classifier function.

In section 2, the mathematics involved are presented, and in section 3 these ideas are applied to the thermophysical invariance problem where the equation $f(\vec{z}) = 0$ is a conservation statement. Finally, some of the theory is confirmed by experimental data and future directions are discussed.

2 Elements of Lie Group Analysis

We explain the theory of Lie Group Analysis as applied to an equation of the form

$$f(\vec{z}) = 0 \tag{1}$$

where $\vec{z} = (z_1, \ldots, z_n) \in \Re^n$ and f is a differentiable function, $f \in C^1(\Re)$. Denote the set of roots of f by

$$V(f) \equiv \{\vec{z} \in \Re^n : f(\vec{z}) = 0\}. \tag{2}$$

If the differential $df \neq 0 \;\; \forall \vec{z} \in V(f)$ then f implicitly defines a manifold. We assume this manifold to be connected[1]. Lie group analysis will determine continuous symmetries only; if the manifold is not connected discrete symmetries may exist and cannot be determined by the methods considered here. An example of a discrete symmetry is reflection. In the physical applications we consider in object recognition problems, discrete

*For extended development of the concepts in this paper contact any of the authors.

[1] A manifold M is connected if to each pair of points in M there exists a curve in M connecting the two points.

symmetries are not an issue. The variables under consideration vary continuously.

The concepts and theory given here can be extended to deal with differential equations - and this is where Lie group analysis is used most often. The generalization of these techniques to differential equations is not difficult. See Olver [1993] for such a treatment.

In general, Lie group analysis is applicable for *systems of equations*, however, any system of equations $g_i = 0$ for $i = 1, \ldots, m$ can be replaced by a single equation $f \equiv \sum_{i=1}^{m} g_i^2 = 0$ in the sense that $V(g_1, \ldots, g_m) = V(f)$. Hence there is no loss of generality in assuming only one equation.

2.1 Curves and Groups of Transformations

A *curve* in $\Re^n$ is a differentiable function

$$\begin{aligned} \varphi &: I \mapsto \Re^n \\ &: \varepsilon \mapsto (\alpha_1, \ldots, \alpha_n) \end{aligned}$$

where $I \subseteq \Re$ is an open interval and $\alpha_i \in \Re$ for $i = 1, \ldots, n$. A curve in $V(f)$ is a curve in $\Re^n$ whose image lies in $V(f)$.

If $(\varphi^1, \ldots, \varphi^n)$ is a vector field on $\Re^n$ (so $\varphi^i = \varphi^i(\vec{z})$) then for each fixed ε

$$\varphi_\varepsilon \equiv (\varphi_\varepsilon^1, \ldots, \varphi_\varepsilon^n) \in \underbrace{C^1(\Re^n) \times \ldots \times C^1(\Re^n)}_{n \text{ factors}},$$

so each φ_ε determines a transformation map of $\Re^n$ given by

$$\begin{aligned} \varphi_\varepsilon &: \Re^n \mapsto \Re^n \\ &: \vec{z} \mapsto (\varphi_\varepsilon^1(\vec{z}), \ldots, \varphi_\varepsilon^n(\vec{z})). \end{aligned}$$

As ε varies over I this determines a family of transformations $\{\varphi_\varepsilon\}_{\varepsilon \in I}$.

If we define the *evaluation function* at $\vec{z}$ as

$$\begin{aligned} e_{\vec{z}} &: \underbrace{C^1(\Re^n) \times \ldots \times C^1(\Re^n)}_{n \text{ factors}} \mapsto \Re^n \\ &: (f_1, \ldots, f_n) \mapsto (f_1(\vec{z}), \ldots, f_n(\vec{z})) \end{aligned}$$

then for a fixed ε,

$$\varphi_\varepsilon(\vec{z}) \equiv e_{\vec{z}}(\varphi_\varepsilon) = (\varphi_\varepsilon^1(\vec{z}), \ldots, \varphi_\varepsilon^n(\vec{z})) \in \Re^n$$

As ε varies over I this determines a curve by

$$\begin{aligned} \varphi_\bullet(\vec{z}) \equiv e_{\vec{z}}(\varphi_\bullet) &: I \mapsto \Re^n \\ &: t \mapsto (\varphi_t^1(\vec{z}), \ldots, \varphi_t^n(\vec{z})). \end{aligned}$$

In this definition $\vec{z}$ is treated as a fixed constant.

As $\vec{z}$ varies over $\Re^n$, $\varphi_\bullet(\vec{z})$ determines a family of curves, $\{\varphi_\bullet(\vec{z})\}_{\vec{z} \in \Re^n}$, one for each point $\vec{z} \in \Re^n$.

The set of transformations $\{\varphi_\varepsilon(\bullet)\}_{\varepsilon \in I}$ has a natural binary operation defined on it given by composition

$$\begin{aligned} \varphi_\varepsilon \cdot \varphi_\delta &: \Re^n \mapsto \Re^n \\ &: \vec{z} \mapsto \varphi_\varepsilon(\varphi_\delta(\vec{z})). \end{aligned}$$

A *group of transformations* $\{\varphi_\varepsilon(\bullet)\}_{\varepsilon \in I}$ is a set of transformations such that the operation of composition satisfies

i. associativity, $\varphi_\varepsilon \cdot (\varphi_\delta \cdot \varphi_\gamma) = (\varphi_\varepsilon \cdot \varphi_\delta) \cdot \varphi_\gamma$
ii. there exist an identity element φ_0, and
iii. each element in $\{\varphi_\varepsilon(\cdot)\}_{\varepsilon \in I}$ has an inverse.

The transformation $\varphi_\varepsilon(\bullet)$ is a parameterized transformation of $\Re^n$. Since it has a single parameter, the group of transformations $\{\varphi_\varepsilon(\bullet)\}_{\varepsilon \in I}$ is called a one-parameter group of transformations.

A one-parameter *Lie Group* is a group which also carries the structure of a 1-dimensional differentiable manifold. This additional structure on a group allows the ability to speak of continuity and differentiability.

2.2 Tangent Vectors and Vector Fields

A *tangent vector* consist of a vector part and a point of application. We denote a tangent vector by $\mathbf{v}_{\vec{z}} = (v_1, v_2, \ldots, v_n)_{\vec{z}}$ where $(v_1, v_2, \ldots, v_n)$ is "the vector part" and $\vec{z}$ is the point of application.

If φ is a curve then $\frac{d}{d\varepsilon}|_{\varepsilon=a}\varphi_\varepsilon$ determines a tangent vector at φ_a.

Each tangent vector $\mathbf{v}_{\vec{z}}$ determines a map by

$$\begin{aligned} \phi_{\mathbf{v}_{\vec{z}}} &: C^1(\Re^n) \mapsto \Re \\ &: f \mapsto \frac{d}{d\varepsilon}|_{\varepsilon=0} f(\vec{z} + \varepsilon \mathbf{v}_{\vec{z}}) \end{aligned}$$

where

$C^1(\Re^n) \equiv$ The set of differentiable functions on $\Re^n$.

For brevity we simply write

$$\mathbf{v}_{\vec{z}}(f) = \frac{d}{d\varepsilon}|_{\varepsilon=0} f(\vec{z} + \varepsilon \mathbf{v}_{\vec{z}})$$

It is an easy exercise to show that $\mathbf{v}_{\vec{z}}(f) = \frac{d}{d\varepsilon}|_{\varepsilon=0} f(\vec{z} + \varepsilon \mathbf{v}_{\vec{z}}) = \frac{d}{d\varepsilon}|_{\varepsilon=0} f(\varphi(\varepsilon))$ for any curve φ through the point $\vec{z}$ satisfying $\frac{d}{d\varepsilon}|_{\varepsilon=0}\varphi^i(\varepsilon) = \mathbf{v}^i$.

Lemma 1 *Let* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *be a vector field and* $f \in C^1(\Re^n)$. *Then*

$$\mathbf{v}(f) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial \vec{z}_i}.$$

Proof: Apply the chain rule.

By this lemma it is meaningful to write

$$\mathbf{v}(f) = (v_1 \frac{\partial}{\partial \vec{z}_1}, v_2 \frac{\partial}{\partial \vec{z}_2}, \ldots, v_n \frac{\partial}{\partial \vec{z}_n}) f$$

where

$$\mathbf{v} = v_1\frac{\partial}{\partial \vec{z}_1}, v_2\frac{\partial}{\partial \vec{z}_2}, \ldots, v_n\frac{\partial}{\partial \vec{z}_n}.$$

Thus a tangent vector, and therefore vector fields as well, can be viewed as either an ordered *n–tuple* or as an operator. It is this ability to view tangent vectors (vector fields) from both perspectives that makes them so powerful.

2.3 Killing Fields and Infinitesimal Generators

The set of vector fields over $\Re^n$ consisting of elements

$$\mathbf{v} = (v_1, v_2, \ldots, v_n)$$

where

$$v_i = v_i(\vec{z}) \in C^1(\Re^n).$$

form a module over the ring $C^1(\Re^n)$ with scalar multiplication being componentwise. Since

$$\begin{aligned} g\mathbf{v} &: C^1(\Re^n) \mapsto C^1(\Re^n) \\ &: f \mapsto g\mathbf{v}(f) \end{aligned}$$

where

$$\begin{aligned} (g\mathbf{v})f &: \Re^n \mapsto \Re \\ &: \vec{z} \mapsto (g(\vec{z})\mathbf{v}_{\vec{z}})f \end{aligned}$$

the set of all vector fields satisfying

$$\mathbf{v}(f) = 0$$

form a submodule since

$$\mathbf{v}(f) = 0 \ \& \ \mathbf{s}(f) = 0 \Rightarrow (\mathbf{v}+\mathbf{s})f = \mathbf{v}(f) + \mathbf{s}(f) = 0$$

and

$$\mathbf{v}(f) = 0 \text{ and } g \in C^1(\Re^n) \Rightarrow (g\mathbf{v})f = 0.$$

The elements of this submodule are called the *killing fields* of f. (In more standard terminology, these elements are *annihilators*. The descriptor "killing fields" is more telling of there role and will be employed here.) A collection of basis elements for this submodule are called *infinitesimal generators*.

Since the infinitesimal generators form a basis for the killing fields of f, every vector field $\mathbf{v}$ such that $\mathbf{v}(f) = 0$, with infinitesimal generators $\{\eta^1, \ldots, \eta^{n-1}\}$, can be written uniquely as

$$\mathbf{v} = \sum_{i=1}^{n-1} g^i \eta^i$$

for some $g^i \in C^1(\Re^n)$ for $i = 1, \ldots, n-1$

2.4 Computation of Groups of Transformations from the Infinitesimal Generators

Groups of transformations can be calculated from the infinitesimal generators by the following

Theorem 2 *If $\varphi_\bullet(\vec{z})$ is a curve in $V(f)$ and $\mathbf{v}$ is a vector field satisfying $\frac{d\varphi^i_\varepsilon(\vec{z})}{d\varepsilon} = v^i(\varphi_\varepsilon(\vec{z}))$ for $i = 1, \ldots, n$ then $\mathbf{v}(f) = 0$. Conversely, if $\frac{d\varphi^i_\varepsilon(\vec{z})}{d\varepsilon} = v^i(\varphi_\varepsilon(\vec{z}))$ for $i = 1, \ldots, n$, $\varphi_\bullet(\vec{z}) = \vec{z} \in V(f)$ and $\mathbf{v}(f) = 0$ then $\varphi_\bullet(\vec{z})$ is a curve in $V(f)$.*

The process of solving the equations to determine a group of transformations determined by the vector field $\mathbf{v}$ is called *the process of exponentiation.*

$$\frac{d\varphi^i_\varepsilon(\vec{z})}{d\varepsilon} = v^i(\varphi_\varepsilon(\vec{z})) \qquad \varphi^i_0(\vec{z}) = \vec{z}_i$$

for $i = 1, \ldots, n$.

Corollary 3 *Let $\mathbf{v}$ be a vector field satisfying $\mathbf{v}(f) = 0$. Each infinitesimal generator of $\mathbf{v}$ determines a curve in $V(f)$.*

Corollary 4 *Let $\{\varphi_\varepsilon\}_{\varepsilon\in\Re}$ be a group of transformations of $V(f)$ determined by the process of exponentiating. If $f(\vec{z}) = 0$ then $f(\varphi_\varepsilon(\vec{z})) = 0$.*

The conclusion of Corollary 4 is really just a tautology since a group of transformations of $V(f)$ means if $\vec{z} \in V(f)$ then $\varphi_\varepsilon(\vec{z}) \in V(f)$.

2.5 The Group of Symmetries, $S_{V(f)}$

We have observed that for an infinitesimal generator η^i of a vector field $\mathbf{v}$ satisfying $\mathbf{v}(f) = 0$, the solution to

$$\frac{d\varphi_\varepsilon(\vec{z})}{d\varepsilon} = \eta^i(\varphi_\varepsilon(\vec{z})) \qquad \varphi_0(\vec{z}) = \vec{z}$$

determines a group of transformations. If $g^i \in C^1(\Re^n)$ then $g^i\mathbf{v}^i$ is a vector field such that $g^i\mathbf{v}^i(f) = 0$ and the solution to

$$\frac{d\varphi_\varepsilon(\vec{z})}{d\varepsilon} = g^i\eta^i(\varphi_\varepsilon(\vec{z})) \qquad \varphi_0(\vec{z}) = \vec{z}$$

determines a curve in $V(f)$, and hence a group of transformations of $V(f)$. More generally, since the infinitesimal generators $\{\eta^1, \ldots, \eta^m\}$ form a basis for the vector fields $\mathbf{v}$ satisfying $\mathbf{v}(f) = 0$, then for any collection of functions

$$g^i \in C^1(\Re^n) \qquad i = 1, \ldots, n-1$$

it follows

$$\left(\sum_{i=1}^{n-1} g^i\mathbf{v}^i\right) f = 0$$

so the solution to the system of differential equations

$$\frac{d\varphi_\varepsilon(\vec{z})}{d\varepsilon} = \left(\sum_{i=1}^{n-1} g^i\eta^i\right)(\varphi_\varepsilon) \qquad \varphi_0(\vec{z}) = \vec{z}$$

determines a curve in $V(f)$, and hence a group of transformations of $V(f)$.

The set of all such transformations determined by this equation is the group of symmetries of $V(f)$, denoted by $S_{V(f)}$. Clearly it is the smallest group containing all of the groups "generated" by the infinitesimal generators $\{\eta^1, \ldots, \eta^{n-1}\}$ as subgroups. Furthermore any transformation of $V(f)$ can be determined by solving such a system of equations.

2.6 Invariant Functions and their Calculation

Suppose we are given the equation $f = 0$. Let

$$\begin{aligned} \Gamma \; &: \; S_{V(f)} \times V(f) \mapsto V(f) \\ &: \; (\varphi_\varepsilon, \vec{z}) \mapsto \varphi_\varepsilon(\vec{z}) \end{aligned}$$

be the $S_{V(f)}$–action on $V(f)$. Then $S_{V(f)}$ acts on $hom(V(f), \Re)$ in a natural way

$$\begin{aligned} \hat{\Gamma} \; &: \; S_{V(f)} \times hom(V(f), \Re) \mapsto hom(V(f), \Re) \\ &: \; (\varphi_\varepsilon, \Phi) \mapsto \varphi_\varepsilon * \Phi \end{aligned}$$

where

$$\begin{aligned} \varphi_\varepsilon * \Phi \; &: \; V(f) \mapsto \Re \\ &: \; \vec{z} \mapsto \Phi(\varphi_\varepsilon(\vec{z})). \end{aligned}$$

Definition 5 *An element $\Phi \in hom(V(f), \Re)$ is an $S_{V(f)}$–invariant of $hom(V(f), \Re)$ if Φ is invariant under the action of $S_{V(f)}$ on $hom(V(f), \Re)$. In other words, the stabilizer of Φ is $S_{V(f)}$*

$$\{\varphi_\bullet \in S_{V(f)} : \varphi_\bullet * \Phi = \Phi\} = S_{V(f)}$$

It is an elementary exercise in algebra to show

Theorem 6 *Let $S_{V(f)}$ be a group acting on a set $hom(V(f), \Re)$. An element $\Phi \in hom(V(f), \Re)$ is an absolute $S_{V(f)}$–invariant of $hom(V(f), \Re)$ if and only if*

$$\Phi(\varphi_\varepsilon(\vec{z})) = \Phi(\vec{z}) \qquad \forall \varphi_\varepsilon \in S_{V(f)}.$$

Proof. See [Arnold *et al.*, 1997].

This necessary and sufficient condition is often taken as the definition of an absolute invariant function. Though the definition of an invariant element of the set $hom(V(f), \Re)$ should be expressed in terms of the more fundamental action

$$\begin{aligned} \Gamma \; &: \; S_{V(f)} \times V(f) \mapsto V(f) \\ &: \; (\varphi_\varepsilon, \vec{z}) \mapsto \varphi_\varepsilon(\vec{z}). \end{aligned}$$

The following theorem gives a necessary and sufficient condition for such an absolute invariant function.

Theorem 7 *Let η^i for $i = 1, \ldots, n-1$ be the infinitesimal generators for the killing fields of f. Then $\Phi \in hom(V(f), \Re)$ is an absolute $S_{V(f)}$-invariant function if and only if $\eta_i(\Phi) = 0$ for $i = 1, \ldots, n-1$.*

Proof. See [Arnold *et al.*, 1997].

3 Lie group analysis in Object Recognition

Several attempts at recognizing object material types using thermophysical invariance theory have been reported recently. Lie group analysis has been applied to each of the different models, including the true differential form found in previous papers [Michel *et al.*, 1997]. The following example began with the formulation presented in [Nandhakumar *et al.*, 1997], in which the radiation term was linearized and embedded into h. Further modifications (discussed below) simplified the Lie group analysis.

$$f \equiv W\alpha\cos\Theta + h(T_\infty - T_s) + K\frac{T_s - T_{int}}{\Delta y} = 0. \quad (3)$$

This model does not contain the energy storage term present in the previous models. Removal of this term allows the conservation statement to become a conservation of heat flux statement as opposed to the conservation of energy statement used before. A key reason for this fundamental shift is to find a model where the terms are independent.

A thorough analysis of the invariants of equation (3) requires the application of Lie group analysis. Consider the conservation equation (3) modeled algebraically by

$$y_1 + y_2\, y_3 - y_2\, a_1 + a_2\, \frac{a_1 - y_4}{y_5} = 0 \qquad (4)$$

where

a_1	$\equiv$	T_s	Surface temperature
a_2	$\equiv$	k	Thermal conductivity
y_1	$\equiv$	$W\ \alpha\ \cos\Theta$	Solar absorption
y_2	$\equiv$	h	Heat transfer coefficient
y_3	$\equiv$	T_∞	Ambient temperature
y_4	$\equiv$	T_{int}	Internal temperature
y_5	$\equiv$	Δy	Depth into the material (along the path of conduction)

The a_i variables are measurable (or guessed) in a recognition scenario and the y_i variables are not. Ideally we would like to find a function of the a_i variables which is an invariant.

In general, W can not be measured, while $\alpha\cos\Theta$ can be estimated. However, for the experiment discussed in the next section the entire term, $W\alpha\cos\Theta$, is measured. Also, a_2 and y_5 are constant, therefore we will have 4 transformation groups after using the equations presented in Section 2.

Generator	Transformation
v_1	$y_1 \mapsto y_1 + \varepsilon$ $a_1 \mapsto a_1 + \frac{y_5}{y_2 y_5 - a_2}\varepsilon$
v_2	$y_2 \mapsto (y_2 - \frac{a_2}{y_5})e^{\varepsilon} + \frac{a_2}{y_5}$ $a_1 \mapsto (a_1 - y_3)e^{-\varepsilon} + y_3$
v_3	$y_3 \mapsto y_3 + \varepsilon$ $a_1 \mapsto a_1 + \frac{y_2 y_5}{y_2 y_5 - a_2}\varepsilon$
v_4	$y_4 \mapsto y_4 + \varepsilon$ $a_1 \mapsto a_1 + \frac{-a_2}{y_2 y_5 - a_2}\varepsilon$

Table 1: Infinitesimal Generators and the corresponding Groups of Transformations. Note: the variables not listed under a Transformation Group undergo the identity transformation. All these transformations are global Lie groups.

The only function invariant under all the transformation groups is

$$\Phi = g[y_1 + y_2\ y_3 - y_2\ a_1 + a_2\ \frac{a_1 - y_4}{y_5}] \qquad (5)$$

where g is an arbitrary function. Hence, analytically, there are <u>no non-trivial</u> invariant functions for (3)! It remains to be determined if additional constraints can be found empirically such that useful quasi-invariants can be found.

4 Experimental Validation of the Group of Transformations

To check the groups of transformations found in the above application, experimental data from a thermocouple data collection performed at Wright-Patterson Air Force Base was used to determine the transformation from one data point A, to another data point B. The "ground truth" data consisted of temperature measurements acquired from thermocouples implanted in various types of materials placed in an outdoor scene and collected over a period of 2 weeks in mid-November. The collection includes varying weather conditions and has extensive records of the atmospheric pressure, ambient temperature, lighting conditions, etc. Multiple temperature measurements of sod, clay, gravel, concrete, asphalt, and aluminum were recorded every 15 minutes and provide rough estimates of all the variables in the conservation equation.

Currently, we measure and estimate all the parameters except h. Although we could also estimate h, we currently derive it from the other estimates and the conservation statement. We plan on estimating h in the future, but for this example, we found it was more useful to derive h for two reasons

1. We can check for reasonable bounds on h to verify when our model is working correctly.
2. By forcing the conservation equation to be true at each time, the transformation groups are better illustrated.

Once we have formed data points for each material at various instances in time, we can verify that our transformation groups work by solving for each ε and applying it to the surface temperature (using the appropriate transformation). If the transformations form a group (as they should), the conservation equation will hold before and after each step. By applying each of the four transformations, we can move between any two points in the group.

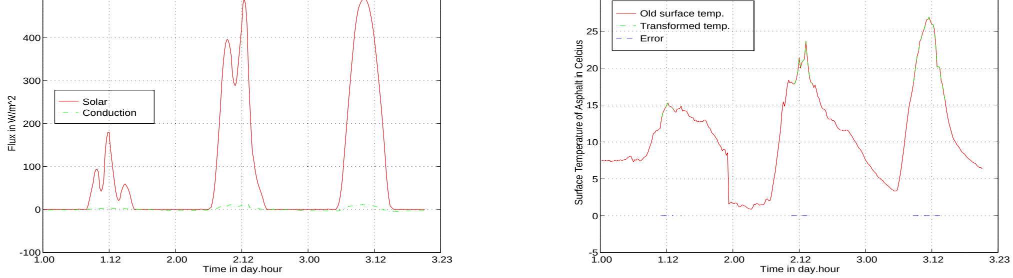
The missing parts of figure 1 correspond to times when the physics-based model was determined to break down. We removed these points for now since the model is not yet robust enough to consider all the different methods of heat transfer. As the model is improved, we will be able to show results for all times and include other factors such as rain, shadows, and transpiration. Ideally an extended time period of data will be used for classification since the material characteristics may be masked at any point by transient or induced effects. Only after collecting an extended period of data could one feel confident in a determination of the materials being viewed.

As previously discussed, we forced the conservation statement to hold by solving for h at each point. If h is estimated, then the resulting conservation statement will not be exactly zero, say $f(\vec{z}) = \delta$. The elements of the group of symmetries would then satisfy $f(\varphi(\vec{z})) = \delta$. A classifier would be designed to determine the threshold for which a point is considered in the class or outside the class. This is similar to the hypothesize and verify scheme suggested in previous papers [Nandhakumar *et al.*, 1997]. However, since we can not measure all these parameters, and since we have shown non-trivial invariants do not exist, we need to look for new formulations of the model and/or quasi-invariants.

5 Discussion

5.1 New physics-based models

Another area of research is the model of the conservation equation. The current model was derived to characterize "typical" data, with no claim



(a) (b)

Figure 1: (a) 3 days of the solar radiation and conduction terms are shown. During the day, solar heating is clearly a dominant effect. (b) The surface temperature of asphalt before and after a 2 hour transformation is shown. The 2nd curve is shifted back 2 hours to show the exact correspondence with the original temperature, thus validating the Lie group analysis.

that it is totally accurate or complete. The model needs extensive revision and validation in order to accomplish 2 major goals

1. to include all common materials in any state (day/night, rain/shine, etc.)
2. to find a model which is both accurate and for which non-trivial invariants exist.

Since the current model clearly does not fully characterize all of the data all of the time, this will be our next step. However, it is likely that model manipulations will not reveal the absolute invariants we desire. Therefore, we must also continue research into ways of finding quasi-invariants.

5.2 Quasi-invariants

From section 2.5 it was determined that any curve in $V(f)$ must satisfy the differential equation

$$\frac{d\varphi_\varepsilon(\vec{z})}{d\varepsilon} = \left(\sum_{i=1}^{n-1} g_i \nu_i\right)(\varphi_\varepsilon) \qquad \varphi_0(\vec{z}) = \vec{z} \tag{6}$$

By curve fitting experimental data the vector fields $\frac{d\varphi_\varepsilon(\vec{z})}{d\varepsilon}$ can be determined. Since the vector fields ν_i for $i = 1, \ldots, n-1$ are known analytically, the scalar coefficients $g_i \in C^1(\Re^n)$ for $i = 1, \ldots, n-1$ can be determined. If (empirically) there is an absolute invariant then at least one of the coefficients g_i would have to be zero. This would imply they lie in a subspace of the module determined by the infinitesimal generators. This could be the result of "overlooking" some physical constraint that is not accounted for by our single equation modeling the problem – the conservation of energy equation. (One known condition we are ignoring are any bounds on the variables.) Furthermore the requirement that any curve satisfy (6) can be used to determine "quasi" (slowly varying) invariants using elementary functional analysis. Locally, if for normalized infinitesimal generators, the condition

$$||g^i||_\infty < \delta \tag{7}$$

for some i is satisfied then a function $\Phi(\bullet)$ can be determined such that $||\frac{d\Phi(\varphi_\varepsilon(\vec{z}))}{d\varepsilon}|| < \delta$. These types of invariants could be just as useful in practice as an absolute invariant.

6 Summary

The techniques of Lie group analysis provide a powerful tool for determining absolute invariant functions which can serve as classifier functions for object recognition problems. We have applied this analysis to the thermophysical invariants problem and we have proven there are no (nontrivial) absolute invariant functions for this model.

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